IMTech Newsletter

Issue 1, June 2021

The first six months

Contents

Editorial

This is the first Newsletter of **IMTech**. It has been born with the assignment to provide a quality communication channel addressed to individual and institutional recipients, with emphasis on the UPC/barcelonatech ecosystem. It aligns with the **IMTech** vision and mission and it is steered by an editorial committee formed by eight members **EC**. The plan is to publish two issues each year, in June and December.

We take notice of two Events related to the participation in the **8ecm** of the **IMTech** members Xavier Cabré as plenary speaker, and Eva Miranda as invited speaker.

In the Interviews section we have collected conversations with the director, Marc Noy, about **IMTech** and his vision for the future; with Jezabel Curbelo, a new Ramón y Cajal researcher at the DMAT and **IMTech**; and with Carme Torras, just distinguished with the 2020 National Research Award "Julio Rey Pastor".

There are three outstanding contributions in the Research Focus. The first is by Albert Atserias and it concerns his joint work with Moritz Müller in which they established that "Automating resolution is NP-hard", a paper published in the Journal of the ACM. The second is by Xavier Cabré and fundamentally it deals with his recent paper "Stable solutions to semilinear elliptic equations are smooth up to dimension 9", coauthored by Alessio Figalli, Joaquim Serra and Xavier Ros-Oton, and published in Acta Mathematica. And the third, by Robert Cardona, Eva Miranda and Daniel Peralta-Salas is about the paper "Constructing Turing complete Euler flows in dimension 3" that just appeared in the Proceedings of the National Academy of Sciences.

The Chronicles section is meant to report on results of recent past activities. In this issue, we cover two fronts. One concerns the remarkable involvement of UPC researchers with problems posed by Covid-19. The other is about the **IMTech** Seminar held on April 15 to celebrate the contributions of László Lovász and Avi Wigderson on the occasion of having been awarded the Abel Prize 2021. The lectures were delivered by the **IMTech** members Oriol Serra and Albert Atserias, respectively, and we include summaries of their contributions. We also include a contribution by Oriol Serra on the work of Lovász on graph limits.

The NL also cares about popular presentations of recent results and they are grouped in the Outreach section. By order of reception, you can read about How to visit two million stars in the shortest time, by Juanjo Rué, and on Front propagation in networks: from the brain to the world, by Alain Goriely (professor of Mathematical Modelling at Oxford and member of the **IMTech** Scientific Advisory Board).

The echoes of recent PhD thesis, as indicated by oustanding publications, are to be found in the section PhD highlights, which on this occasion refer to the theses Active superelasticity in three-dimensional epithelia of controlled shape, by Ernest Latorre, and Arithmetic applications of the Euler systems of Beilinson-Flach elements and diagonal cycles, by Óscar Rivero.

To look at current advances in mathematics and its applications we devote the section Reviews. On this occasion, it focuses on Mathematics and Machine Learning. Six of the selected items are books: *Large Networks and Graph Limits* (by László Lovász) and *Mathematics and Computation: A Theory Revolutionizing Technology and Science* (by Avi Wigderson), the winners of the Abel Prize 2021 and cited in the section Chronicles; *Reinforcement Learning of Bimanual Robot Skills* (by Adrià Colomé and Carme Torras); *Data-Driven Computational Neuroscience: Machine Learning and Statistical Models* (by Concha Bielza and Pedro Larrañaga); *Geometric Deep Learning: Grids, Groups, Graphs, Geodesics, and Gauges* (a monograph by Michael M. Bronstein, Joan Bruna, Taco Cohen, and Petar Veličković); and *A thousand brains: A new theory of intelligence* (by Jeff Hawkings). And nine items are papers, all readily accessible as preprints through the inserted links.

As backmatter we have a section for Quotations, a page Contact on the Editorial Committee, and an alphabetical Index for names an concepts in this issue.

The design of the graphic motive featured in the cover, and in the image below, is due to Maria Alberich. It plays with the UPC logo and Hilbert's curve.

Events

?Xavier Cabré, Plenary speaker at the **8ecm** (20-26 June 2021, Slovenia, online conference).

Doctorate in Mathematics under the direction of Louis Nirenberg at the Courant Institute of Mathematical Sciences, with postdoctoral training at the Institute for Advanced Study at Princeton and at the Laboratoire Jacques-Louis Lions at the Université Pierre et Marie Curie (Paris 6), ?Xavier Cabré (Barcelona, Spain, 1966) is ICREA Research Professor since 2003 at the UPC and Full Professor of Applied Mathematics at the same university since 2008.

?Xavier Cabré was a Harrington Faculty Fellow at the University of Texas at Austin (2001-02) and is a Fellow of the AMS (Inaugural Class 2013).

His field of research is that of elliptic and parabolic Partial Differential Equations (PDEs) and, in particular, their connections with problems in Differential Geometry, Optimal Stochastic Control, and in diffusion processes in Physics and Ecology. He co-authored, together with Luis Caffarelli, an already classic reference in PDEs [1]. His results include: an improved version of the Alexandroff-Bakelman-Pucci estimate [2]; the Krylov-Safonov inequality for diffusion processes in Riemannian varieties [3]; his proof in 3D of the De Giorgi's conjecture on phase transitions and minimal surfaces [4]; a new proof of the classical isoperimetric inequality [5]; the gradient estimate for non-local minimal graphs [6]; and the regularity of stable solutions of semi-linear elliptic equations up to dimension nine [7].

[1] Caffarelli-Cabré-1995 [2] Cabré-1995 [3] Cabré-1997 [4] Ambrosio-Cabré-2000 [5] Cabré-2000, Cabré-2008, Cabré-RosOton-Serra-2013 [6] Cabré-Cozzi-2019 [7] Cabré-Figalli-RosOton-Serra-2020

EUROPEAN MATHEMATICAL SOCIETY

?Eva Miranda, Invited speaker at the **8ecm** (20-26 June 2021, Slovenia, online conference).

Full Professor in Geometry and Topology at UPC and **IMTech** member, ?Eva Miranda is attached to CRM, and *Chercheur affilié* at the Observatoire de Paris. She is the director of the Laboratory of Geometry and Dynamical Systems at UPC and the group leader of the UPC Research group GEOMVAP.

Her research lies at the crossroads of Differential Geometry, Mathematical Physics and Dynamical Systems, [1]. Her main areas of expertise are Symplectic and Poisson Geometry [2], and integrable systems with group actions accounting for their symmetries [3]. Her interests include the geometrical and dynamical aspects of the singularities arising in Celestial Mechanics and Fluid Dynamics, and in Symplectic and Poisson manifolds, as well as with mathematical models for their quantization [4].

Her current research focuses on the singular Weinstein conjecture [5], and the extension of the Floer complex to Poisson manifolds; on the study of Fluid Dynamics through a geometrical mirror; and on singularities in Symplectic Geometry, which includes generalizations of b-symplectic manifolds, desingularization theorems, integrable systems, and the role of singularities in different branches such as KAM theory and quantization theory. For stationnary Euler flows, for instance, the geometrical mirror is a contact mirror and in this context a special mention is deserved by the recent construction [6] of a theoretical fluid computer.

She has been distinguished with an ICREA Academia Prize in 2016 and a *Chaire d'Excellence* of the ***Fondation Sciences Mathématiques de Paris*** (2017-2018). She is also an active member of the mathematical community as a member of international scientific panels and committees. She served as a member of the Scientific Advisory Board of the CRM and since 2021 she is a member of the Scientific Committee of RSME. Since May 2018 she is a member of the Governing Board of the Barcelona Graduate School of Mathematics and since 2020 she is a member of the *Conseil d'Administration* de l'**Institut Henri Poincaré** in Paris. For more information, see 8ECM: An Interview of Eva Miranda (10 May 2020).

[1] Miranda-Zung-2004 (arXiv).
Miranda-Monnier-Zung-2012 (arXiv).
[2] Guillemin-Miranda-Pires-2014 (arXiv).
Kiesenhofer-Miranda-2016 (arXiv),
Kisenhofer-Miranda-Scott-2016 (arXiv),
Delshams-Miranda-Kiesenhofer-2017 (arXiv).
[3] Guillemin-Miranda-Pires-Scott-2015 (arXiv),
Esposito-Miranda-2017 (arXiv).
[4] Miranda-Presas-2015 (arXiv),
Guillemin-Miranda-Weitsman-2018 (arXiv).
Mir-Miranda-2021 (to appear in AMP)
[5] Miranda-Oms-2020, Miranda-Peralta-Oms-2020.
[6] Cardona-Miranda-Peralta-Presas-2020
Cardona-Miranda-PeraltaSalas-Presas-2021 (arXiv)
Cardona-Miranda-Peralta-2021

Interviews

?Marc Noy (**IMTech** director)

Born in Barcelona (1958), he is Full Professor at UPC since 2003, and his main fields of research are combinatorics and graph theory. His recent works focus on asymptotic enumeration and random graphs, and connections with logic. In previous years he also worked on discrete geometry, Tutte polynomials and classical enumerative combinatorics. He has played an active role in the development of Discrete Mathematics in Spain, through the training of researchers and the organization of conferences and advanced courses. He has received several distinctions, including the Humboldt Research Award (2012), the Von Neumann Chair from the Technical University of Münich (academic year 2012-13), and the medal Narcís Monturiol (2018) from the Catalan Government. He has been a plenary speaker at numerous international conferences and an invited speaker at the Seoul ICM-2014. He was director of the Barcelona Graduate School of Mathematics (2015-2018) and PI of the excellence grant BGSMath/María de Maeztu (2015-2019). He is currently head of the research group GAPCOMB (Geometric, Algebraic and Probabilistic Combinatorics), and coordinator of the European project RandNET, Randomness and learning in networks (2021-2024).

NL. ***IMTech* has been opened the last of a rather long row of mathematical institutes in Spain. Could you summarize the main reasons that advocate for its launching?**

Research in mathematics and related areas has reached an excellent level at UPC in the past decades. The Institute is intended to provide international visibility to UPC researchers and to encourage collaboration among them. At the same time, it provides the right academic structure for organizing scientific activities, promoting interdisciplinary research, and building strategic alliances. It is our goal to collaborate with national and international centres in the organization of joint activities and the participation in joint projects.

Could you outline the gestation and birth of IMTech?

It has been a long process, initiated about five years ago by Professor ?Oriol Serra, then chair of the Department of Mathematics. The project gained momentum with the addition to the project of the School of Mathematics and Satistics, and the departments of Statistics and Operations Research, Computer Science, and Civil and Environmental Engineering. In our academic environment, creating a new research institute is not an easy matter and one needs to gather consensus and convince the corresponding authorities. By the end of 2019 we got the approval of the UPC Rector and Council, and in October 2020 the approval from the Catalan Government. We started working right away at the beginning of 2020 with a "shadow" steering committee so by October we already had an operational web page and appointed an international Scientific Advisory Board.

What policies do you envisage to boost the development of IMTech in the next few years?

Our main goal is to become a reference centre in mathematics in the European scene. I am convinced that we have the potential for doing so given that our faculty includes internationally recognized researchers leading very competitive research groups. However we need urgently to incorporate new senior and junior members to **IMTech** in order to boost high-quality research. Given that our research system suffers from chronic under-funding, increasing funding both public and private is an absolute priority. As a first step **IMTech** will provide technical support for improving the success rate of its members in competitive calls.

Which should be the role of IMTech in mathematics outreach from your point of view?

Due to a lack of scientific tradition in our country, Spanish (and Catalan) governments have never made science a priority and well into the 21st century the science and innovation budget is less than half the average of the European Union. Increasing public awareness of the key role of science in our society could help reversing this situation. We are witnessing such a change with the pandemic: the public is well aware that the astounding speed at which effective vaccines have been produced has been possible only by a tremendous and coordinate effort of top-level science. Given the diversity of **IMTech** (mathematics and statistics, computer science, engineering) we are well positioned to reach the public and illustrate the role of science and technology through our research. We aim at having a significant impact through media and the organization of events addressed to students and the society at large. Finally we intend to involve our PhD students and postdocs in dissemination as part of their global training.

How do you think that IMTech could help to engage more women to pursue PhD studies in mathematics and its applications?

This is a goal in which all actors must be involved, including of course **IMTech**. Our members have a very active participation in several UPC doctoral and master's programs in mathematics and related areas, where they can support female students by encouraging them to pursue research and acting as tutors and supervisors.

?Jezabel Curbelo

In 2020 she received the Antonio Valle Prize to Young Researchers, awarded by the SEMA. In 2015, the ?Donald L. Turcotte Award of the American Geophysical Union Nonlinear Geophysics Group "to recent PhD recipients for outstanding dissertation research that contributes directly to the field of nonlinear geophysics", and the ?Vicent Caselles Award of the RSME–BBVA Foundation granted to young outstanding PhD's. Recently she has been distinguished with the L'Oréal-Unesco award For Women In Science, the first given in mathematics in Spain. Web UPC news.

Born in Tenerife (Los Realejos, 1987), she received her BS in Mathematics (2009) from the ULL, a MSc in Mathematics and Applications (2010) and a PhD in Mathematics (2014) from UAM. At present she is a ?Ramón y Cajal researcher at the UPC's DMAT and **IMTech**.

In early January 2021, the **EC** held the following interview with her.

NL. ***What was your motivation to study a PhD?***

I wanted to find a job when I finished my degree (I was going to finish in 2009, and the crisis had already started), so during my last degree year I applied to everything that I could while I was doing the CAP (teaching certificate to teach in high school). At that time, predoctoral grants included the master's year, in my opinion better than now, and I got it even before I graduated. So I didn't have time to think. I started the Master and when I thought about the possibility to study a PhD I had already started doing it.

Do you have any recommendations for students starting a PhD in mathematics?

Read every single day!

Could you please describe your academic career and the influence of the different research environments you have experienced?

I've been very lucky. The work environment that has surrounded me in all the places I have been during my academic career has been unbeatable, especially thanks to my colleagues, and I am sure that thanks to that I have arrived where I am now.

As an undergraduate student, I was awarded with a collaboration grant in Jorge Betancor's group at the ULL, where I received my BS in mathematics. Thanks to JAE-Intro, I met ?Ana M. Mancho (CSIC - ICMAT) who would later become my thesis supervisor at ICMAT. I was a PhD student in the JAE-Predoc fellowships program and Teaching Assistant (Profesor Ayudante) at UAM. I was a LabEX LIO postdoc at Laboratoire de Géologie de Lyon and Juan de la Cierva Formación Postdoc at UPM, followed by a post of Assistant Professor (Profesor Ayudante Doctor) at UAM. I have also visited UCLA and the ENS-Lyon several times. Now, I am Ramón y Cajal researcher at UPC.

You are currently working on several applications to the oceans and the atmosphere. Could you please tell us about the role of mathematics in climate challenges?

Mathematics plays a fundamental role in climate challenges working with a set of complex equations that simulate the Earth's behaviour. For instance, to unravel the causes of climate change and make predictions about the future climate a weather and climate modelling and forecasting is required. And they have a strong mathematical base. We use numerical methods, stochastic processes, PDE analysis, nonlinear dynamics, statistical diagnosis, signal processing, time series analysis, data filtering and assimilation, ... Mathematics are a valuable tool to squeeze useful information out of the available observation data and the mathematical simulations help us to play with different scenarios in order to estimate the possible consequences of specific actions.

What problems would you like to see solved during this decade?

The Navier–Stokes existence and smoothness problem.

How do you deal with the computational and simulation aspects of your work?

I do the codes locally and I perform small tests, or simple calculations on my laptop or desktop computer. For the rest, I remotely connect to clusters or supercomputers in different parts of the world.

How do you see the academic career for young mathematicians?

Young mathematicians are promising, they have enthusiasm, new and good ideas and they do research of high quality and excellence. But, the early stages of their academic career are fraught with difficulty. For instance, in Spain there is a lack of means, opportunities, financing and stability. The sight of Science is lost and paperwork takes its place, because the time is spent applying for jobs or funding.

How can we have more women involved in mathematics and have them as role models?

This is an important and difficult question. Initiatives like 11deFebrero.org for the International Day of Women and Girls in Science, databases of women scientists such as AMIT, equality plans in universities and research centers, programs like #steMatEsElla or women's associations, as AWM and EWM, are helpful, but there is still a lot of work to do.

?Carme Torras Research Professor at the Spanish Scientific Research Council (CSIC).

She received MSc degrees in Mathematics and Computer Science from the Universitat de Barcelona and the University of Massachusetts, respectively, and a PhD degree in Computer Science from the Technical University of Catalonia (UPC-BnaTech). She is IEEE Fellow, EurAI Fellow, member of Academia Europaea, member of the Reial Acadèmia de Ciències i Arts de Barcelona, and of the Institute of Catalan Studies (IEC), and a recipient of the Narcís Monturiol medal from the Government of Catalonia "for outstanding contributions to the scientific and techological progress" and the "Julio Rey Pastor" National Research Award 2020 (for her "pioneering contributions in the area of intelligent robotics and social robotics", the first woman to be awarded this recognition). Former Editor of the IEEE Transactions on Robotics, she has published books and articles on neural models, computer vision, artificial intelligence and robotics, areas in which she has led sixteen European projects, including an ERC Advanced Grant on the robotic manipulation of garments, Clothilde.

NL. ***You have been awarded the National Research Award for research. First of all, our warmest congratulations! The first questions are about the research system in Spain. How do you see the Spanish research system presently? What do you think are the most pressing changes Spain needs in its research policy?***

The research quality in Spain is uneven, but my perception is that in some areas we do not have much to envy to the top players in the European Union. I mean scientifically. As regards to all other aspects of the research ecosystem we have a lot to envy: the social prestige of the research career, which implies better salaries and favours the promotion, attraction and retention of talent; policies committed to excellence, which take risks prioritizing the most promising fields and projects after a serious study; companies that value I+D and invest locally; policies enforcing public-private partnership; less bureaucracy, reducing the fine-grained economic evaluation of projects and deepening their scientific evaluation; stabilizing research programs and calls by decoupling them from yearly economic budgets and election terms; faster decisions and lots of technical support to free researchers of the high load of administrative matters. All this requires much higher funding, of course, but also high-level coordination to avoid duplicating efforts, and moving from a model "researcher does it all" to "research is done in a team" including technical, administrative, and managerial support.

What would be your recommendation to a talented young researcher who has finished a PhD in Spain and wishes to pursue an academic career?

My recommendation would be to look for a postdoc position abroad in a group with the highest possible reputation and within a prestigious project (a European project with interesting partners, an ERC grant...). The Euraxess platform lists many such offers. The ELLIS PhD and Postdoc Program offers also the possibility of belonging to a community of postdocs sharing research interests and opportunities. According to my experience, it is more likely to come back and pursue an academic career in Spain after a postdoc in Europe than in the US, where young researchers are quickly offered high salaries in companies. Moreover, a postdoc in a European group opens the possibility to continue collaborating in the future through a European project. It is much harder to get funding to continue collaborating with a group in Japan or the US, for example.

Now we turn to the connections of your research with mathematics. Your bachelor's degree was in mathematics at the Universitat de Barcelona. How this influenced your research career? And which memories do you have from that time?

I have wonderful memories of very enlightening conversations in that magnificent building. Its symmetric structure with the two cloisters, one for mathematics and the other for literary studies, and the impressive library (and the bar!) strategically placed in the corridor joining them... it was a perfect picture of the synergy between the two types of knowledge I was interested in. Actually, I enrolled in math and philosophy at the time, and I remember talking about ?Nietzsche's *Thus Spoke Zarathustra* and, without any solution of continuity, turning to passionate discussions about ?Borges' poems or ?Cortázar's *Rayuela*. I owe a lot to my classmates and friends at that time. Actually, I have kept in contact with many of them, whom I have met at UPC and the SCM, among other places. Having studied at UB has also helped to organize shared seminars in mathematical topics relevant to robotics and computer vision, as well as to attract talented mathematical graduates to our group.

After graduating in mathematics you went to UMass for a Master's degree in computer science. How was your experience at UMass?

Well, before going to UMass, I worked for one year in a company, Texas Instruments. It was a nice experience, especially to make me realize that this was not what I wanted. Thus, I wrote a letter to Prof. ?Michael Arbib, a mathematician and author of a book that had impressed me: *Brains, Machines and Mathematics*. To my surprise and delight, he answered inviting me to pursue a master degree at UMass. And there I went. Again, it was a wonderful experience. I followed a Computer Science program with intensification in Brain Theory (Prof. ?Arbib's speciality). I learned a lot on artificial and natural intelligence, and I made lots of good friends. For the second time, a new world opened in front of my eyes, and once more there was a very well assorted library on campus! There I found the book *The Passions* by the philosopher ?Robert Solomon, which many years later inspired my novel *La mutació sentimental* (*The Vestigial Heart*, in the English version published by MIT Press).

At which point in your career did you decide to make robotics your research field? How did that happen?

When I finished my MSc degree, I faced a dilemma: returning to Barcelona or staying in the US. It was a tough decision and I hesitated a lot and even went back and forth to UMass a couple of times before I finally prioritised my personal life over academia and definitely returned. I looked for places where to continue my research on mathematical models in neuroscience, but I couldn't find any. The Ramón y Cajal hospital in Madrid offered me a position as a technician, where I could help a top research group led by Dr. ?García-Austt and Dr. ?Buño, but I could not develop a research career there if I was not a medical doctor or a biologist. Then, Prof. ?Gabriel Ferraté offered me a research position at the Cybernetics Institute at UPC on a new field that was just starting: robotics. Luckily, I was able to apply the mechanisms of temporal-pattern learning and motor neurocontrol that I studied in my PhD thesis to robot perception and adaptive motion control.

What is the role of mathematics in your current research? What areas of mathematics are more relevant?

Mathematics has always played a key role in my research, but the relevant areas have changed a lot along the years. My PhD thesis was published in Lecture notes in biomathematics, a label that encompassed mathematical modelling and simulation, dynamical systems, and numerical analysis applied to biological entities. When I turned to robotics, I continued applying modelling and simulation, but also affine and projective geometry (in relation to computer vision) and combinatorial topology (e.g., cell complexes to stratify the configuration space of parallel robots). Other works I did in robotics are more related to discrete mathematics, graphical models, probabilistic algorithms, and computational geometry. In our most recent project, CLOTHILDE, we are combining techniques from computational topology (persistent homology, topological invariants like the linking number or writhe) with machine learning algorithms, i.e., statistical pattern recognition and optimization.

One last question: What do you think should be the priorities for a new institute like IMTech?

IMTech's website says that the goals of the Institute are to enhance high-quality research, interdisciplinary collaboration and knowledge and technology transfer, international talent attraction, and fundraising for supporting UPC researchers. If you are asking what mechanisms should be implemented to reach these goals and how priorities should be assigned... this is a very tough question! In my opinion, **IMTech** is a good platform to establish synergies between different groups at UPC, to favour talk and collaboration across mathematical areas, and make more visible the mathematical potential of the university. Thus, activities to promote this gathering should be prioritized (work on common problems, seminars, open challenges, etc.). If **IMTech** becomes more than the sum of its part(ner)s, attainment of the goals above will follow naturally.

Interviews in the media:

LaVanguardia-20201212
Vilaweb-20201226
FullsEnginyeria-20210107
SalaPremsaUPC-20201113
MagazineDigital-20180902

?Carme Torras and the robot Clothilde

Research focus

Proof Search and the Structure of Solution Spaces

by ?Albert Atserias (IMTech, UPC). *Received 9 Jan, 2021*

In 2019, ?Moritz Müller and I resolved a longstanding open question on the computational complexity of proof search. The preliminary version of the article "Automating Resolution is NP-Hard" was the co-winner of the Best Paper Award at the 60th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2019). In its final version, the paper was invited for submission to the Journal of the ACM, and published in 2020 [1]. Why does the proof that *yet another problem is NP-hard* deserve such an important award? In this note I try to explain why. Let me begin with the historical background.

This year is the 50th anniversary of ?Steve Cook's article "The Complexity of Theorem Proving Procedures" [5]. Cook's seminal work established that the satisfiability problem for propositional logic is NP-complete. For the first time, the generic task of searching a solution within an exponentially big space of candidate solutions was reduced to that of a concrete, well-defined, and combinatorially simple-to-state problem. The importance of Cook's Theorem was quickly recognized. Only one year later, ?Richard Karp wrote another landmark paper where he established that "a large number of classic unsolved [computational] problems of covering, matching, packing, routing, assignment and sequencing are equivalent". All of them, Karp showed, are equivalent to the problem of recognizing the satisfiable formulas of propositional logic [7].

The fact that hundreds, if not thousands, of NP-complete problems were discovered since then raises an intriguing question. Might any of these equivalent problems admit sufficient structure to allow for efficient search algorithms? Might the beautiful theory of matchings on graphs be useful to find Hamilton cycles? Might the powerful tools of field theory help in solving systems of polynomial equations? In a sense, what the theory of NP-completeness says is that any attempt to give computationally useful structure to the search spaces of such problems is bound to fail. Or at least fail as much as it has failed for the problem of determining truth in the algebra of propositions since its conception by ?George Boole more than one and a half centuries ago [3].

Proof theory is the branch of mathematical logic that links the syntactic concept of proof with the semantic notion of truth. A formula is unsatisfiable if and only if its negation is a tautology, and tautologies have proofs in any one of the many complete proof systems for propositional logic. Thought this way, the proof theory of propositional logic can be seen as giving structure to the satisfiability problem. Just like an assignment that satisfies a formula is a certificate that a solution exists, a proof of its negation is a certificate that a solution *does not* exist. The trouble is that while truth assignments are always short, proofs could be exponentially longer. But what if we focus on short proofs? Can reasonably-sized proofs be found in reasonable time? This is the question of *automatability* of a proof system. In his 1971 article, Cook asked to study "the complexity of theorem proving procedures", including Resolution, which is the object of our 2019 article, and the rest of this note.

A formula in conjunctive normal form, a CNF, is a conjunction of a set of clauses, where a clause is a disjunction of variables or negations of variables. If each clause has at most k literals, then we call it a k-CNF. For example, $(p \vee q \vee r) \wedge (\neg p \vee \neg q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg r)$ is a 3-CNF with three variables and four clauses. Resolution is a proof system to establish the unsatisfiability of CNFs by deriving new clauses from old. The goal is to end up proving the empty, trivially unsatisfiable clause. Such proofs are called *refutations*. The single inference rule of Resolution cannot be simpler: from two clauses of the form $A \vee x$ and $B \vee \neg x$, derive the clause $A \vee B$. It is obvious that if an assignment satisfies the two premises in the inference, then it must also satisfy the conclusion. In particular, if the empty clause can be proved, then the initial formula must be unsatisfiable. It is an entertaining exercise to show that the converse is also true in quantitative terms: if a CNF is unsatisfiable, then the empty clause can be proved in no more than 2^n inference steps, where n is the number of variables. Needless to say, if shorter proofs exist, then those would be preferable since a proof of length 2^n, even for $n = 500$, would not fit in the observable universe.

Unlike the space of satisfying assignments of a formula, which appears unstructured, it is a remarkable fact that the space of *short* Resolution proofs appears to have non-trivial structure. An important result due to ?Ben-Sasson and ?Wigderson [2] states that if a 3-CNF formula with n variables has a Resolution refutation of length s, then it also has one in which all its clauses have $O(\sqrt{n \log s})$ many literals. For polynomially small s, the number of such clauses is weakly exponential, i.e., of the form $\exp(O(n^{1/2}(\log n)^{3/2}))$, which means that an algorithm of this run-time exists that finds a proof. In summary: when polynomially short proofs exist, weakly exponentially long ones can be found.

Valid as it is, this result begs the question whether polynomial-size Resolution proofs are common at all. If they were a very rare exception, then the statement that short proofs can be found would be essentially void. With this perspective in mind, it is at least worrying that the early results on the proof complexity of Resolution seemed to indicate that polynomial-size Resolution proofs were indeed an exception. An influential result of ?Chvátal and ?Szemerédi, titled "Many Hard Examples for Resolution" [4] shows that, in a well-defined average-case sense, *most* unsatisfiable 3-CNF formulas are indeed exponentially hard for Resolution. Are we then flirting with the void?

One first answer to this is that just like the worst-case model for computational complexity has its weaknesses, any average-case model has its own problems. The formulas that appear in applications are not only not worst-case; even less are they drawn from the probability distribution for which they were shown exponentially hard. The spectacular success of so-called SAT-solvers, software packages that are able to find satisfying assignments or Resolution refutations of many formulas with many variables, illustrates the point. Consider the so-called Boolean Pythagorean Triples Problem: Can every initial segment $1, \ldots, n$ of the natural numbers be partitioned in two in such a way that all triples of the form $a^2 + b^2 = c^2$ are avoided in each part? In a major breakthrough, in 2016 a SAT-solver was able to show that the CNF formula that encodes the problem with n variables is satisfiable if, and only if, $n \leq 7824$ [6]. This solved a Ramsey theory open problem posed by ?Graham in the 1980s. Despite being referred to as "Two-hundred-terabyte maths proof is largest ever" [8], it should be noted that, for $n = 7825$, the proof must be able to rule out 2^{7825} many partitions of $1, \ldots, 7825$. A 2×10^{14} bytes long proof is, in comparison, *microscopically* tiny.

A second answer to the question whether short Resolution proofs are common or rare exceptions, can be found in our 2019 result. I claim that this is, indeed, what makes it surprising. Formally, we say that Resolution is *automatable* if there is an algorithm that, given an unsatisfiable formula, finds a Resolution refutation in time polynomial in the length of the shortest refutation. After a long series of works on the theme going back two decades, we finally proved that the problem is NP-hard: Resolution is *not* automatable unless P = NP. Perhaps more importantly, concerning our question on the abundance of short proofs, here is what the result has to offer. As it goes, the NP-hardness proof establishes that every formula F can be efficiently converted into a new one, G, whose polynomially short Resolution refutations *correspond* to the satisfying assignments of F. More precisely, if F is satisfiable, then G is unsatisfiable and has a polynomially short Resolution refutation that can be easily constructed from a satisfying assignment of F. In contrast, when F is unsatisfiable, the technically hardest part of the NP-hardness proof shows that G does not even have weakly exponentially longer Resolution refutations. Among other consequences, this means that short Resolution proofs not only abound; despite exhibiting a certain amount of non-trivial structure, they are able to mirror the search space of solutions of *any* problem in NP.

Only a handful of other problems in NP without polynomial-time algorithms are known whose natural solution spaces are sufficiently structured to allow better-than-exponential algorithms. Perhaps the two most famous examples are the problem of factoring integers into primes, and the graph isomorphism problem. None of those, however, is even expected to be NP-hard. The problem of finding polynomially short Resolution proofs stands out, then, as a notable exception.

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Regularity of stable solutions to reaction-diffusion equations up to dimension 9 (after [1])

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The paper [1] solves a semilinear version of Hilbert's XIXth problem that remained open since the 1970s. In dimensions $n \leq 9$, it establishes the smoothness of local minimizers of the functional

$$A(u) = \int_{\Omega} \left(\frac{1}{2} |\nabla u(x)|^2 - F(u(x)) \right) dx, \tag{0.1}$$

where Ω is an open set of $\mathbb{R}^n$, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, and $F : \mathbb{R} \to \mathbb{R}$ is a given function. Making a first variation $u + \varepsilon v$, integrating by parts, and defining $f = F'$, one easily sees that critical points of the functional satisfy the *reaction-diffusion equation* or semilinear elliptic equation

$$-\Delta u = f(u) \quad \text{in } \Omega \subset \mathbb{R}^n. \tag{0.2}$$

The setting is relevant since, in numerous physical phenomena and geometric problems, observable states try to minimize a certain functional. In classical mechanics, and also in (0.1), the functional is called *action* and it is the integral of a Lagrangian. In geometry, two important examples are geodesics in a Riemannian manifold (where the functional is their length) and minimal surfaces in Euclidean space (where the functional is area). In all these problems, one considers minimizers —or more generally, critical points— of the functional among functions, or surfaces, with prescribed boundary values (for instance, the end points of the geodesic).

In Hilbert's XIXth problem the functional is given by $A(u) = \int_{\Omega} L(\nabla u(x))\, dx$ for some convex function $L : \mathbb{R}^n \to \mathbb{R}$ and, therefore, A is convex. As a consequence, if it admits a critical point, this will be an absolute minimizer which accordingly fulfils the *principle of least action* in mechanics. However, for many potentials F in (0.1), as well as for minimal surfaces, the functional is not convex and the states that one observes in nature are only local minimizers (minimizers among small perturbations). Such critical points are *stable*, in the sense that the second variation of the functional at such state is nonnegative definite.

A situation in which this occurs concerns catenoids: the soap film or minimal surface formed between two coaxial parallel circular rings. In [2], catenoids are experimentally produced in a lab and photographed while the distance between the two circular wires increases. For a range of distances, the observed catenoids are minimal surfaces which are stable but not absolute minimizers, since the two flat disks spanned by the wires have less area.

In the 1960s the Italian school proved that the Simons' cone $x_1^2 + \ldots + x_m^2 = x_{m+1}^2 + \ldots + x_{2m}^2$ is an absolute minimizer of area in $\mathbb{R}^{2m}$ if $n = 2m \geq 8$. At the same time, a sequence of outstanding contributions (J. Simons' being a prominent one) established the smoothness, when $n \leq 7$, of every minimal hypersurfaces of $\mathbb{R}^n$ that is an aboslute minimizer. It is a long-standing open problem to extend this regularity result to the larger class of *stable minimal surfaces*. It is only known to hold for surfaces in $\mathbb{R}^3$. See the survey [3] for more details.

The paper [1] takes on the analogue question, the regularity of *stable solutions*, for the reaction-diffusion equation (0.2). The problem was motivated by a combustion model for the thermal self-ignition of a chemically active mixture of gases in a container Ω. Here, $u = u(x)$ is the temperature of the point $x \in \Omega$. For a large class of nonlinearities, which include the so called *Gelfand problem* $-\Delta u = \lambda e^u$ arising from *Arrhenius law* in chemical kinetics, the situation is similar to the one of catenoids. Indeed, for a certain range $\lambda \in (0, \lambda^*)$ of parameters, there exists a stable solution (that is, a solution at which the action has a nonnegative definite second variation) which is not an absolute minimizer (since the action functional is unbounded from below). In terms of the associated nonlinear heat equation $v_t - \Delta v = \lambda e^v$, it corresponds to the stationary temperature of the container observed for very large times when the initial temperaure is constant $v(\cdot, 0) = 0$ and ignition fails. When $\lambda \geq \lambda^*$ (successful ignition), no solution exists —in the same way that catenoids did not exist for large distances between the wires.

When $n \geq 3$, $\Omega = B_1$ is the unit ball,

$$u = \log \frac{1}{|x|^2}, \quad \text{and} \quad f(u) = 2(n-2)e^u,$$

a simple computation shows that we are in the presence of a singular solution of (0.2) vanishing on ∂B_1. Similarly to the Simons' cone in minimal surfaces theory, this solution turns out to be stable in high dimensions, precisely when $n \geq 10$. On the other hand, in the 1970s Crandall and Rabinowitz [4] established that, for $f(u) = e^u$ or $f(u) = (1+u)^p$ with $p > 1$, stable solutions in any smooth bounded domain Ω are bounded (and hence smooth, by classical elliptic regularity theory) when $n \leq 9$. This result was the main reason for Haïm Brézis to raise the following question in the 1990s (which we cite almost literally from a later reference):

Brezis [5, Open problem 1]: *Is there something "sacred" about dimension 10? More precisely, is it possible in "low" dimensions to construct some f (and some Ω) for which a singular stable solution exists? Alternatively, can one prove in "low" dimensions that every stable solution is smooth for every f and every Ω?*

The last twenty five years have produced a large literature on the topic; see the monograph [6]. The main developments proving that stable solutions to (0.2) are smooth (no matter what the nonlinearity f is) were made

- by Nedev [7] in 2000, when $n \leq 3$ (and f is convex);
- by the author and Capella [8] in 2006, when $\Omega = B_1$ (u is radially symmetric) and $n \leq 9$;
- by the author [9] in 2010, when $n \leq 4$ (and Ω is convex).

Note that the 2006 result in the radially symmetric case, [8], accomplished the optimal range $n \leq 9$ for every nonlinearity f. This gave hope for the result to be true also in the general nonradial case, though no certainty was assured. The work [1] finally solves Brezis' open problem:

Theorem *In dimensions $n \leq 9$, under the only requirement for the nonlinearity f to be nonnegative, every (energy) stable solution of (0.2) is smooth in the open set Ω.*

Furthermore, adding the vanishing boundary condition $u = 0$ on $\partial\Omega$, the article proves regularity up to the boundary when Ω is of class C^3 and $n \leq 9$, assuming now f to be nonnegative, nondecreasing, and convex. Both results come along with new universal Hölder-continuity estimates:

$$\|u\|_{C^\alpha(\overline{B}_{1/2})} \leq C\|u\|_{L^1(B_1)} \quad \text{and} \quad \|u\|_{C^\alpha(\overline{\Omega})} \leq C_\Omega \|u\|_{L^1(\Omega)},$$

where $\alpha \in (0,1)$ and C are dimensional constants, while C_Ω depends only on Ω. These estimates are rather surprising for a nonlinear problem, since they make no reference to the reaction nonlinearity f. The stability of the solution is crucial for their validity. For the expert reader, [1] also establishes another open problem, now from [10]: an apriori $H^1 = W^{1,2}$ estimate for stable solutions in all dimensions n.

If the nonnegativeness of f is a needed requirement in the theorem remains as an open question. It is only known to be unnecessary for $n \leq 4$, as well as for $n \leq 9$ in the radial case.

An analogue result for equations involving the p-Laplacian has been proved in 2020 by the author, Miraglio, and Sanchón, [11]. In terms of dimensions it is optimal for $p > 2$, but not for $p < 2$. On the other hand, for the recently very active area of fractional Laplacians, an optimal result is largely open —even in the radial case. The optimal dimensions for regularity have only been accomplished in a 2014 work of Ros-Oton [12] for the Gelfand nonlinearity $f(u) = \lambda e^u$ in domains which are symmetric and convex with respect to all coordinate directions.

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Undecidable fluid particle paths and 3D fluid computers (after [1]).

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Introduction

In the book *The Emperor's new mind* [10], ?Sir Roger Penrose returns to the artificial intelligence debate to convince us that creativity cannot be presented as the output of a "mind" representable as a Turing machine. This idea, which is platonic in nature and highly philosophical, evolves into more tangible questions such as: *What kind of physics might be non-computational?*

The ideas of the book are a source of inspiration and can be taken to several landscapes and levels of complexity: Is hydrodynamics capable of performing computations? (Moore [9]). Given the Hamiltonian of a quantum many-body system, is there an algorithm to check if it has a spectral gap? (this is known as *the spectral gap problem*, recently proved to be undecidable [5]). And last but not least, can a mechanical system (including a fluid flow) simulate a universal Turing machine (*universality*)? (Tao [15–17]).

This last question has been analyzed in relation to the conjecture of the regularity of the Navier-Stokes equations [14], which is one of the unsolved problems in the Clay's millennium list. In [18] Tao speculates on a connection between a potential blow-up of the Navier-Stokes equations and Turing completeness and fluid computation. It is interesting to mention that another of the one million dollars problem on the same list whose resolution is still pending is the P *versus* NP *problem*, which concerns the complexity of systems. Grosso modo, the question is whether any problem whose solution can be *verified* by an algorithm polynomial in time ("of type NP") can also be *solved* by another algorithm polynomial in time ("of type P"). The delicate distinction between verification and solution has opened up an intricate scenery combining research in theoretical computer science, physics and mathematics. Although there is no apparent relation between the two celebrated problems in Clay's list, understanding a fluid flow as a Turing machine may shed some light on their connection.

On the other hand, undecidability of systems is everywhere and also on the invisible fine line between geometry and physics: As proven by ?Freedman [21], non-abelian topological quantum field theories exhibit the mathematical features (combinatorics) necessary to support an NP-hard model. This relates topological quantum field theory and the Jones polynomial (as described by Witten [19]) to the $P \neq NP$ problem. Other undecidable problems on the crossroads of geometry and physics are the stability of an n-body system [8], the problem of finding an Einstein metric for a fixed 4-fold as observed by ?Wolfram [20], ray tracing problems in 3D optical systems [12], or neural networks [13]. Fundamental questions at the heart of low dimensional geometry and topology such as verifying the equivalence of two finitely specified 4-manifolds [20], or the problem of computing the genus of a knot [2], have also been proven to be undecidable and NP-hard problems, respectively.

In [1] we addressed the appearance of undecidable phenomena in fluid dynamics and proved the existence of stationary solutions of the Euler equations on a Riemannian 3-dimensional sphere that can simulate any Turing machine (i.e., they are Turing complete). These solutions describe the dynamics of an inviscid and incompressible fluid in equilibrium. The type of flows that we consider are Beltrami fields, a particularly relevant class of steady Euler flows. Our novel strategy fusions the computational power of symbolic dynamics with techniques from contact topology and its connection with hydrodynamics unveiled by ?Sullivan, ?Etnyre and ?Ghrist more than two decades ago [6].

Euler equations and Beltrami fields

Euler equations model the dynamics of an incompressible fluid flow without viscosity. Even if they are classically considered on $\mathbb{R}^3$, they can be formulated on any 3-dimensional Riemannian manifold (M, g) and follow the same mnemonics as the classical ones from vector calculus (for an introduction to the topic see [3, 11]). The equations can be written as:

$$\begin{cases} \frac{\partial}{\partial t}X + \nabla_X X &= -\nabla p\,, \\ \operatorname{div} X = 0\,, \end{cases}$$

where p stands for the hydrodynamical pressure and X is the velocity field of the fluid (a non-autonomous vector field on M). Here $\nabla_X X$ denotes the covariant derivative of X along X. A solution to the Euler equations is called stationary whenever X does not depend on time, i.e., $\frac{\partial}{\partial t}X = 0$.

The Euler equations can be defined in higher dimensions [3], an extension that is very useful to show that the steady Euler flows exhibit remarkable universality features as we proved in [4]. In this note, however, we shall restrict ourselves to 3-dimensional fluids.

Two fundamental notions

- ❑ A volume-preserving (autonomous) vector field X on M is **Eulerisable** if there exists a Riemannian metric g on M compatible with the volume form, such that X satisfies the stationary Euler equations on (M, g):

$$\nabla_X X = -\nabla p\,, \qquad \operatorname{div} X = 0\,. \tag{0.1}$$

- ❑ A divergence-free vector field X on (M, g) is called **Beltrami** if

$$\operatorname{curl} X = fX\,,$$

with $f \in C^\infty(M)$. The classical Hopf fields on the 3-sphere $\mathbb{S}^3$ and the ABC flows on the 3-torus $\mathbb{T}^3$ are examples of Beltrami fields [11].

Turing machines

A Turing machine is a mathematical model of a theoretical device manipulating a set of symbols on a tape following some specific rules. It receives, as input data, a sequence of zeros and ones and, after a number of steps, returns a result, also in the form of zeros and ones. More concretely: A Turing machine is defined as $T = (Q, q_0, q_{\text{halt}}, \Sigma, \delta)$, where Q is a finite set of states, including an initial state q_0 and a halting state q_{halt}, Σ is the alphabet, and $\delta : (Q \times \Sigma) \longrightarrow (Q \times \Sigma \times \{-1, 0, 1\})$ is the transition function. The input of a Turing machine is the current state $q \in Q$ and the current tape $t = (t_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$.

How a Turing machine works. If the current state is q_{halt} then *halt the algorithm* and return t as output. *Otherwise compute* $\delta(q, t_0) = (q', t'_0, \varepsilon)$, replace q with q', t_0 with t'_0 and t by the ε-shifted tape.

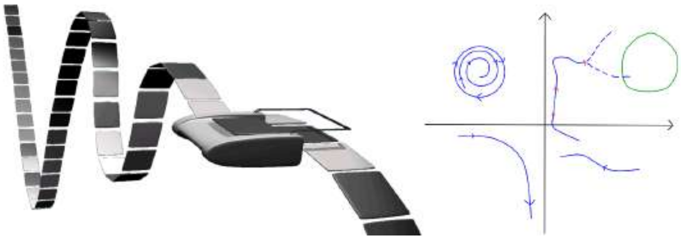
The **space of configurations** is denoted by $\mathcal{P} := Q \times \Sigma^{\mathbb{Z}}$. The *global transition function* $\Delta : Q \setminus \{q_{\text{halt}}\} \times \Sigma^{\mathbb{Z}} \to \mathcal{P}$ (it sends a configuration in $\mathcal{P}$ to the configuration obtained after applying a *step of the algorithm*). A Turing machine is *reversible* if the global transition function Δ is *injective*.

Native to the realm of computer science, the notion of Turing completeness in the title of the article [1] refers to a system that can simulate any Turing machine.

The halting problem. In computability theory, the halting problem is the problem of determining, from a description of an arbitrary computer program and an input, whether the program will stop running (halting state), or continue to run forever. Alan Turing proved in 1936 that a general algorithm to solve the halting problem for all possible program-input pairs cannot exist. A key part of the proof is the formulation of a mathematical definition of a computer and program (the preceding notion of Turing machine), which allowed to prove that the halting problem is undecidable. The halting problem is historically important as it was one of the first problems to be proved undecidable.

Turing machines and universality. An Eulerisable field on M is Turing complete if it can simulate any Turing machine. In other words, the halting of any Turing machine with a given input is equivalent to a certain bounded trajectory of the field entering a certain open set of M.

Alan Turing



Turing machine and a Turing complete vector field associated to a point and an open set.

A Turing machine can be simulated by a dynamical system (a vector field or a diffeomorphism).

Our construction

The construction that we provide in [1] can be understood as a mathematical theoretical fluid computer; it takes as input data a point in space, processes it –following the trajectory of the fluid through that point– and its outcome is the next region to which the fluid has moved. In the proof we use techniques from geometry, topology and dynamical systems developed over the last 30 years. Specifically, we combine symplectic and contact geometry and fluid dynamics with computer science theory and mathematical logic.

Geometry to the rescue. Beltrami fields have a strong geometrical flavour. In terms of the dual one-form $\alpha = \iota_X g$, and the volume form μ, **the equation of Beltrami fields** reads

$$\begin{cases} f\iota_X\mu = d\alpha \\ d\iota_X\mu = 0 \end{cases}$$

When f does not vanish, the one-form α is a contact form (i.e., such that $\alpha\wedge d\alpha \neq 0$) as observed longtime ago in [6]. One of the main characters of contact manifolds and the study of their dynamics are Reeb vector fields R, which are characterized by the conditions $\alpha(R) = 1$ and $\iota_R d\alpha = 0$. In [6] it was also proved that any Reeb vector field defines a steady Euler flow for some ambient metric. This correspondence gives a mirror between contact geometry and Beltrami fields.

A key point of our construction lies in the following theorem, which we also prove in [1]:

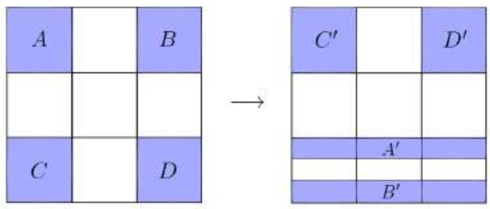
Theorem 0.1 (A Reeb suspension)**.** *Let (M, ξ) be a contact 3-manifold and $\varphi : D \to D$ an area-preserving diffeomorphism of the disk which is the identity on (a neighborhood of) the boundary. Then there exists a defining contact form α whose associated Reeb vector field R exhibits a Poincaré section with first return map conjugated to φ.*

A bit of symbolic dynamics. In 1991, Moore generalized the notion of shift to be able to *simulate any Turing machine.* Given a Turing machine there is a generalized shift ϕ conjugated to it. Conjugation means that there is an injective map $\varphi : \mathcal{P} \to \Sigma^{\mathbb{Z}}$ such that the global transition function of the Turing machine is given by $\Delta = \varphi^{-1}\phi\varphi$.

Key observation. Generalized shifts are conjugated to maps of the *square Cantor set* $C^2 := C \times C \subset I^2$, where C is the (standard) *Cantor ternary set* in the unit interval $I = [0, 1]$.

Point assignment. Take $\Sigma = \{0, 1\}$. Given $s = (...s_{-1}.s_0s_1...) \in \Sigma^{\mathbb{Z}}$, we can associate to it an *explicitly constructible point* in the square Cantor set. We just express the coordinates of the assigned point in base 3: the coordinate y corresponds to the *expansion* $(y_0, y_1, ...)$ where $y_i = 0$ if $s_i = 0$ and $y_i = 2$ if $s_i = 1$. Analogously, the coordinate x corresponds to the *expansion* $(x_1, x_2, ...)$ in base 3 where $x_i = 0$ if $s_{-i} = 0$ and $x_i = 2$ if $s_{-i} = 1$.

Moore proved that any generalized shift is conjugated to the restriction to the square Cantor set of a piecewise linear map of I^2. This map consists of finitely many area-preserving linear components defined on blocks. If the generalized shift is bijective, then the image blocks are pairwise disjoint. Each linear component is the composition of two linear maps: a *translation* and a positive (or negative) power of the *horseshoe map.*



An example of a piece-wise linear map of the square.

In [1] we prove:

Proposition 0.2. *For each bijective generalized shift and its associated map of the square Cantor set ϕ, there exists an area-preserving diffeomorphism of the disk $\varphi : D \to D$ which is the identity in a neighborhood of ∂D and whose restriction to the square Cantor set is conjugated to ϕ.*

This allows us to assemble all the former pieces into the final construction.

End of the proof: On the other side of the mirror. Using the contact mirror, the Proposition above and Theorem 0.1 we can prove the main result in [1]:

Theorem 0.3. *There exists an Eulerisable flow X on $\mathbb{S}^3$ that is Turing complete.*

The metric g that makes X a stationary solution of the Euler equations can be assumed to be the round metric in the complement of an embedded solid torus (which is the region where the universal Turing machine is encoded).

Conclusions

Undecidability and chaos. Because of the undecidability of the halting problem for Turing machines, an important property of a Turing complete dynamical system is the existence of trajectories which exhibit undecidable long-term behavior. Specifically, it is undecidable to determine if the trajectory through an explicit point will intersect an explicit open set of the space.

One of the main consequences of the result is that it allows us to prove that certain phenomena of hydrodynamics are undecidable. That is, there is no algorithm to ensure that a fluid particle will pass through a certain region of space in finite time (in metaphoric terms: if we send a message inside a bottle, we cannot guarantee that it will reach its recipient). This inability to predict, which is different from that established by chaos theory, can be understood as a new manifestation of the turbulent behaviour of fluids.

In chaos theory, unpredictability is associated with the extreme sensitivity of the system to the initial conditions –the flutter of a butterfly can generate a tornado–, whereas we prove that there can be no algorithm that solves the problem, so that it is not a limitation of our knowledge, but of the mathematical logic itself.

Theoretical fluid computers and the Navier–Stokes problem. ?Tao?Tao, T. launched a program in 2016 [14–18] based on the Turing completeness of the Euler equations to address the blow up problem for the Navier-Stokes equations included in the Clay Foundation's list of Millennium Problems. Tao's proposal is, at the moment, speculative. The idea of Tao is to use a Theoretical fluid computer to force the fluid to accumulate more and more energy in smaller regions, until a singularity is formed, that is, a point at which the energy becomes infinite. However, at the moment it is widely open how to do this for the Euler or Navier–Stokes equations.

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Chronicles

Abel Prize 2021

On April 15, shortly after the announcement of the 2021 Abel Prize, **IMTech** organized a seminar on the scientific contributions of the awardees ?László Lovász and ?Avi Wigderson, given respectively by ?Oriol Serra and ?Albert Atserias.

The talks (in Catalan, with slides in English) were recorded and can be viewed here (Serra) and here (Atserias). The ?NL Editorial Team has compiled a summary of the talks. In addition here is a link to the Abel Prize ceremony containing lectures by ?Lovász and ?Wigderson.

?Oriol Serra started his presentation with a short biographical sketch of ?Lovász and then moved to his main scientific contributions, including: matchings in graphs and hypergraphs, the perfect graph theorem, topological combinatorics, the Lovász number, the Lenstra-Lenstra-Lovász lattice basis reduction algorithm, the Lovász local lemma, and graph limits.

Then he focused on three topics. First the LLL algorithm and its impact on Combinatorial Optimization (the ellipsoid method and integer programming), Algebra and Number Theory (efficient factorization of polynomials, diophantine approximation and the disproof of Mertens conjecture), Cryptography, and many more.

He then moved to the Lovász number of a graph and its links with information theory, highlighting the fact that it can be computed in polynomial time using semidefinite programming.

Finally he focused on graph limits, which is the subject of a separate contribution in this section. (Lovász's book on *Large Networks and Graph Limits* is included in the Reviews section.)

?Albert Atserias started his presentation illustrating how the probabilistic method can prove existence results (Ramsey graphs, Boolean circuit complexity) in a non-explicit way.

He then moved to one of ?Wigderson's most celebrated results, namely that randomness in polynomial-time algorithms can be simulated, deterministically, with no more than polynomial overhead, conditioned on the existence of hard explicit functions. He discussed the complexity classes based on randomized algorithms, including the classes RP and BPP, early contributions of Lovász, and those of Nisan-Wigderson, and Impagliazzo-Wigderson in the 1990s. The latter can be rephrased by saying that either randomness is unnecessary (P = BPP) or the Satisfiability problem, the paramount NP-complete problem, can be solved (non-uniformly) in subexponential time.

Then he presented the Nisan-Wigderson pseudorandom generator based on explicit hard functions and combinatorial designs.

He concluded the talk with the topic of Polynomial Identity Testing, where Lovász, Wigderson and others have made fundamental contributions. (?Wigderson's book *Mathematics and Computation* is included in the Reviews section. It can be freely downloaded from his web page. In the folder 'Talks' there you can find slides of the lecture "The Value of Errors in Proofs", which seems very close to the slides of his Abel lecture).

László Lovász and Graph Limits by ?Oriol Serra (**IMTech**, UPC) *Received 31 Mar, 2021.*

The 2021 bestowal of the Abel Prize to ?László Lovász and ?Avi Wigderson acknowledges the strength of combinatorics and computer science in mathematics, a strength which has grown steadily in the last century.

The two awarded mathematicians perfectly illustrate the quoted interplay among the two areas and with the other areas of mathematics. Among their many substantial contributions this short note concentrates on one of the latter ones of ?László Lovász, the development of the theory of Graph Limits, which is representative of his creative mathematical work and its links to combinatorics and computer science.

As a professor at Yale University, ?Lovász accepted in 1996 the proposal to joint the Theory Group of Microsoft Research in Redmond (Washington) to lead a research program aimed at building mathematical foundations for the analysis of large networks. In spite of its growing popularity in the scientific community, the topic of large networks had been usually addressed with a more experimental approach than from a mathematical perspective. There are many scientific and technological motivations behind this interest. The web, the internet, the neurons in the brain or the biological ecosystems constitute networks of billions of nodes which have to be understood and sometimes managed.

Graphs are the natural mathematical model of networks. A graph consists of a pair of sets, a set V of nodes and a set E of pairs of nodes. It expresses in its simpler form a binary relation, which in the network context is connection. Graphs are certainly one of the favorite mathematical objects of ?Lovász. The new challenge was to formulate the appropriate questions when the goal is to concentrate on the large numbers, which may have a completely different nature than the ones usually brought up when dealing with concrete sizes.

The natural choice was to appeal to the old venerable notion of limit and formulate it in the context of discrete structures. This natural idea had been expressed in many ways before ?Lovász undertook his program. Random discrete structures are naturally observed from the asymptotic perspective and many limit objects have been proposed and studied. The area of extremal graph theory, analyzing discrete structures with some forbidden substructure, becomes meaningful only from this asymptotic perspective. Ramsey theory dealing with the inevitable appearance of discrete substructures in large ones bears the record of largest numbers one has to consider in mathematics. A usual tool in these areas, the regularity method of ?Endre Szemerédi (another Abel Prize winner), is by nature only meaningful for very large graphs. What was missing was the very notion of limit object and of convergence which should capture this asymptotic perspective. The theory of graph limits provides such notions.

A *graphon*, the chosen name for the limit object (a contraction of graph and function) is simply a measurable function W from the unit square to the unit interval which is symmetric in its variables. A finite graph with n vertices can be naturally associated with a graphon by means of an equipartition of the unit interval in n subintervals and define the graphon which takes value 1 on the square $I \times J$ of the partition of the unit square whenever the vertices i and j are adjacent and zero otherwise. Graphons also model random graphs where the binary values 0 and 1 are replaced by probabilities. Once the limit object is defined a metric is introduced to provide the appropriate notion of convergence of a sequence of graphs.

To provide the suitable notion the convergence of a sequence of graphs has a twofold interpretation, combinatorial and analytical. From the combinatorial side, for every fixed graph H one measures the probability that a random map from H to some graph G is a graph homomorphism, a map preserving adjacency, and a sequence of graphs is convergent if the corresponding sequence of probabilities converges for every choice of H. The meaning of this notion clearly describes the structure of a graph in terms of its behaviour with respect every finite graph, an idea that suitably captures the notion of a convergent sequence of graphs. From the analytic point of view the notion of *cut distance* is introduced which measures the distribution of the edges in a graph by the maximum density of edges along a cut. Crucially there is a natural extension of this cut distance to the space of graphons for which the resulting topological space is compact. Compactness of this space describes in an elegant way one the key tools in combinatorics and graph theory, the regularity lemma of ?Szemerédi which is in fact proved to be equivalent to it. This is one of the landmarks of the theory.

A technical work now ensues which shows that the two above notions of convergence, the one measuring homomorphisms from any given graph and the one provided by the cut distance, coincide. In particular convergence of a sequence of graphs is equivalent to the Cauchy property, which allows to analyze convergence from the terms of the sequence.

An account of the general theory briefly descibed above is developed in detail along the 470 pages of the ?Lovász' monograph *Large networks and graph limits*, published by the American Matehmatical Society in 2012, which also describes several applications and poses a number of intriguing open questions. Among them the study of the topological nature of graphons which is only partially understood.

Among the many awards and recognitions ?Lovász has received in his mathematical career, he recently received the Hypathia Prize awarded jointly by the city of Barcelona with the Academia Europeae in its first edition. In his speech at the occasion ?Lovász made the following beautiful remark: *it may happen that the study of large networks opens a new view of our world which can be seen as an immense network of interactions.*

Covid-19 and Mathematics at UPC

At the time of this writing, the search in arXiv produces more than 2,500 works with 'covid-19' in the title, and above 500 if we add the word 'model'. Of the latter, about 400 were posted in 2020 and more than 100 in 2021. For a readable general followup of the pandemic, see for example [1]. See also COVID-19: Epidemiology, virology, and prevention.

Our community also engaged in modelling various aspects of the pandemic from early 2020 till the present time, often in collaboration with epidemiologists and other groups elsewhere. Here we summarise the contributions that have had remarkable outcomes (so far) in terms of scientific publications and social impact.

The group BIOCOMSC focused from very early on Covid-19 and by now some of its members (particularly ?Sergio Alonso Muñoz, ?Enric Álvarez Lacalle, ?Daniel López Codina, and ?Clara Prats Soler) are widely known because of the frequent requests of their expertise from the administration and the media. The doctoral students ?Martí Català and ?David Conesa have also been outstanding contributors, as well as over twenty undergraduate students of the UPC degrees in Physical Engineering and in Data Science and Engineering.

The model they have exploited, based on the *Gompertz equation* and classified as Mathematical Epidemiology, is succinctly described in [2] and detailed reports on their activities can be accessed online: BIOCOMSC/Covid-19. See, for instance their Daily report (Analysis and prediction of COVID-19 for EU-EFTA-UK and other countries), which shows in particular the wide impact of their modelling. Among their journal publications, which include clear presentations of the model they work with and their struggles in the treatment of uncertainties, see [3] and [4].

We gratefully acknowledge ?Clara Prats Soler for having provided us with the key facts about the BIOCOMSC research endeavors and for her diligent attention to our requests.

Currently the members of the research group GRBIO (Biostatistics and Bioinformatics, their work focusing "in advanced applications and in the theoretical and computational development of new methodologies") belong to UPC (10), UB (14), and to VHIR (1). Members of the group supported early international expert pronouncements urging the state and autonomous governments for actions concerning lockdown measures.

They have also applied Multi-state models to data provided by HM Hospitals, as summarized in [5]. The paper [6] is part of a special issue of Σ*mathematics* devoted to statistical methods for the analysis of infection diseases and one of its findings was that "an age-specific quarantine policy might be more efficient than a unified one in confining COVID-19".

We are grateful to ?Guadalupe Gómez Melis for having provided us with key data concerning their research on Covid-19 problems.

In February 2020, a team or researchers was formed with the aim to design statistical models for the quantification in real time of the unreported cases of Covid-19 in Spain. Under the name Group UDAT, its members were ?David Moriña, (UAB, BGSMath, Coordinator), ?Alejandra Cabaña (UAB), ?Pere Puig (UAB), ?Amanda Fernández-Fontelo (HU Berlin), and ?Argimiro Arratia (UPC, **IMTech**). With support from the Instituto de Salud Carlos III de Madrid and the Ministerio de Ciencia e Innovación, they developed a discrete model based on Hidden Markov Chains, which allowed to estimate daily new cases of Covid-19, and a continuous hierarchical Bayesian model, which additionally allowed to evaluate the effect of sanitary measures on the evolution of the pandemic. The group also collaborated in the project CEMat: Acción matemática contra el coronavirus by providing their estimates to their meta-model of Covid-19. The publications [7] and [8] were an outcome of those research experiences. For the corresponding software, see the CRAN package Good.

We are thankful to ?Argimiro Arratia for having send us the information for the UDAT chronicle and his prompt answers to our questions.

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Outreach

How to visit two million stars in the shortest time

by ?Juanjo Rué

(NL English translation, with permission from **El País**, of the article *Cómo visitar dos millones de estrellas en el menor tiempo posible* published on December 14, 2020, in the "Café y Teoremas" series)

The Gaia project of the European Space Agency (ESA) has cataloged 2,079,471 stars. Could we find a route to visit them all, in the shortest possible time? Recently, a group of researchers have come up with a path that is very close to the answer. These types of questions are studied in an area of mathematics known as mathematical optimization or mathematical programming, of great importance not only theoretically but also for its applications to real life and for its ramifications in other disciplines.

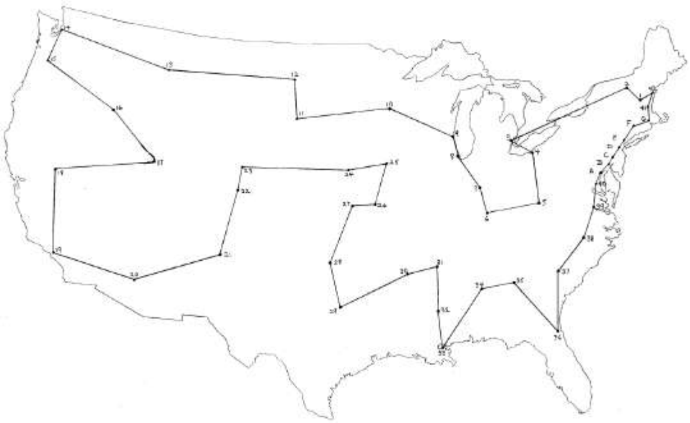
It is about identifying, among all the possibilities to solve a problem, the best one, with respect to an established criterion. We ask ourselves these types of questions when we choose a telephone company (we want to choose the one that offers us a more adequate service at a lower price), when we are going to run errands in our neighborhood (we want to move around spending as little energy and time as possible) or, planning a long walk through the Universe. In all cases, we choose between all the existing possibilities (the set of telephone companies or the various routes through our neighborhood or through the Cosmos) to optimize an amount (money or time).

The problem of the star ride is an example of the *traveling salesmen* problem, one of the most important in the field. It is about a network of cities, interconnected by roads. From it we define a graph, that is, an abstract mathematical structure that contains objects (the vertices of the graph) and relationships between them (the edges of the graph). In our case, the vertices would be the cities, and there would be an edge between two cities if they are connected by a road. In addition, we will include the distance between the connected cities giving a weight to the edges, quoted in kilometers, for example.

Now, a traveler looks for a way to visit all cities without repetition, ending the tour in the initial city and using the fewest kilometers possible. In mathematical language, the question is to find what is called a *Hamiltonian cycle* (a path that begins and ends at the same point that only passes once through each vertex) in the graph under study, in such a way that the sum of weights of its edges is the smallest possible.

For a small number of cities, the account can be done by hand, but the complexity of the problem increases rapidly depending on the cities we are considering. Taking more than 20 cities and assuming that the number of connections is high (the maximum number of roads in this case would be 190, assuming there is a direct road between any pair of towns), the algorithmic problem of calculating all routes becomes an intractable issue. In fact, it is well known that it is a difficult algorithmic problem: finding the solution, even using powerful computers, requires so much computational time that it is considered impractical.

However, modern methods allow finding near-optimal solutions in reasonable calculation times. These are solutions to the problem that do not minimize the number of kilometers, but are very close to it, that is, their difference from the optimum is small. A wide variety of mathematical techniques are used in this search, including the use of heuristics, statistics, probability and analysis.



Original map of the work by ?Dantzig, ?Fulkerson and ?Johnson.

The study of this problem motivated the development of many of the central ideas in this discipline. The fundamental work, from 1954, is *Solution of a large-scale travelling-sales problem*. In it, ?George Dantzig (father of the algorithm), ?Delbert R. Fulkerson and ?Selmer M. Johnson laid the foundations for the formal study of the traveling salesman problem and, incidentally, shaped the entire discipline of mathematical programming (which already received a great boost during World War II). As a curious fact, the authors studied the problem for 48 cities in the United States, and obtained a route to visit them all following a Hamiltonian cycle. The tour length was 12,345 miles.

Currently, and thanks to the algorithmic power of computers, much more complex systems can be studied than in the 1950s and, surprisingly, results very close to the optimal value can be obtained. This is the case of the recent work by ?Bill Cook (University of Waterloo, Canada) and ?Keld Helsgaun (Roskilde

University, Denmark): they have found, using mathematical optimization techniques, the shortest route to visit the 2,079,471 stars indexed by the Gaia project of the ESA (Star Tours). The study, which can be found here, finds a route to visit all these stars in ... 94,208,157.5 light years! And that is also very close to being the optimal solution. This recent and impressive application that shows the mathematical power of the ideas of this discipline.

Front propagation in networks: from the brain to the world

by ?Alain Goriely (Mathematical Institute, University of Oxford). *Received 22 Feb, 2021.*

We live in a connected world where people, goods, information, and diseases travel from one region to the next. In the years of COVID, a particularly dramatic example of this propagation phenomenon is, of course, the transmission of the coronavirus from a single seed location (Wuhan, China), to the rest of the world through an intricate network of local and global travel routes.

The same phenomenon also appears in a much smaller system, the human brain where toxic proteins related to neurodegenerative diseases like Alzheimer's or Parkinson's are believed to originate in a single region and are transported to the rest of the brain through the so-called *connectome*, the network of axonal pathways connecting different regions of the brain [1, 2].

From a mathematical perspective, both phenomena can be understood as the propagation of an autocatalytic process on networks and the main question is to understand its overall dynamics: If a process like a disease starts at a seed location, how long will it take to appear at other locations, and then develop through a full-scale invasion, leading to a global pandemic for a disease or to dementia for the brain?

The simplest model for such invasion process on a network is the discrete equivalent of the celebrated Fisher-Kolmogorov-Petrovsky-Piskunov reaction-diffusion equation (Fisher-KPP) [3]. If $p_i = p_i(t)$ is the quantity of interest in a region i evolving with time t, then on a network with N nodes, it takes the simple form of system on N differential equations

$$\frac{dp_i}{dt} = -\rho \sum_{j=1}^{N} L_{ij} p_j + \alpha p_i (1 - p_i), \quad i = 1, \ldots, N \qquad (0.2)$$

$$p_s(0) = \epsilon \ll 1, \quad p_i(0) = 0, \; i \neq s, \qquad (0.3)$$

where L is the symmetric graph Laplacian that models diffusion process on a graph and encodes the connection between regions ($L_{ij} = 0$ if there is no connection between regions i and j and $\min_{ij}\{L_{ij}\} = -1$.

The system is seeded at node s. Now, consider the particular case when the autocatalytic term is dominant over the diffusion term: $\rho/\alpha \ll 1$. An example of a network with 5 nodes is shown in Fig. 1 together with the dynamics of each variable at different nodes.

A number of interesting observations can be made. Early on, the dynamics of the first node evolves on its own, with all other variables negligible. Then, the invasion starts appearing at other nodes, and very soon, the entire system is invaded. The mathematical question is to find the typical halftime of invasion for each node, represented by the value half-way (the time τ_i when $p(\tau_i) = 1/2$) and an approximation for each curve.

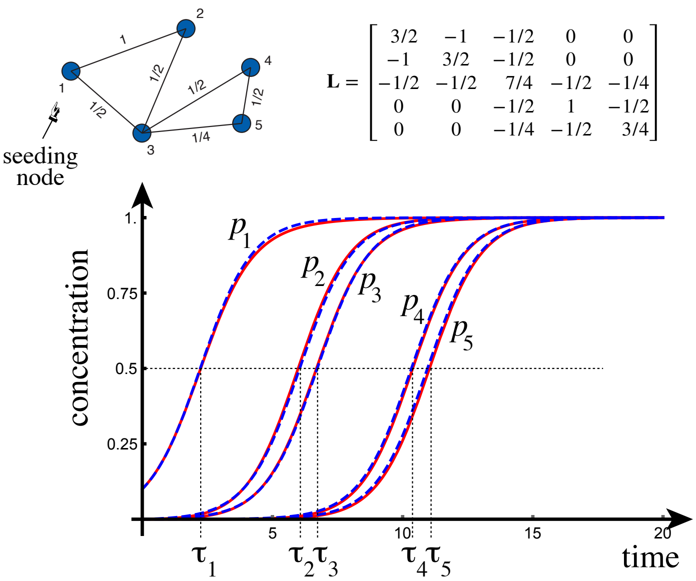


Figure 1: Example of dynamics on a 5-node network. Initially, only the first node is seeded ($p_1(0) = 1/10$). Due to the coupling term, the second node and then all other modes are invaded. The numerical solution is shown in red (solid) and the nonlinear asymptotic solution are in blue (dashed) ($\alpha = 1$, $\rho = 1/100$).

There are two ways to approach the problem. The first one is to isolate two nodes and linearise the sub-system for these two nodes. Then it is not too difficult to show that the halftime between two nodes i and j is given

$$\omega_{ij} = \frac{1}{\alpha} W_0 \left(\frac{\alpha}{\rho L_{ij}} \right), \quad i, j = 1, \ldots, N, \qquad (0.4)$$

where W_0 is the Lambert function. This flight time induces naturally a notion of temporal distance on a graph by finding the shortest Lambert temporal distance Ω_{ij} between nodes i and j based on the node-node distance. It also leads to reasonable estimate for the halftimes for a seed at node s. The initial τ_s is given by isolating that single node and solving the nonlinear system to obtain

$$\widehat{\tau}_s \approx \frac{1}{L_{ss}\rho - \alpha} \log \left(\frac{\epsilon(\alpha - L_{ss}\rho - \alpha\epsilon)}{L_{ss}\rho(2\epsilon - 1) + \alpha(\epsilon - 1)^2} \right),$$
$$\widehat{\tau}_i \approx \tau_s + \Omega_{si}, \; i \neq s. \qquad (0.5)$$

The second method is to find an asymptotic expression of each concentration in terms of sigmoid-like function. Then the problem is to consider the nonlinear interactions of these different localized solutions, a problem that leads to a system of transcendental equations for the times τ_i that can easily be solved numerically by using the approximation $\widehat{\tau}_i$ as a first guess.

We can now scale up that simple example to the entire brain by considering the connectome provided by averaging axonal pathways on multiple brains and leading to the 1015 node connectome shown in Fig. 2. If we seed this connectome with a toxic protein in the entorhinal cortex, where Alzheimer's disease is known to originate, we can follow the evolution of the disease through the brain.

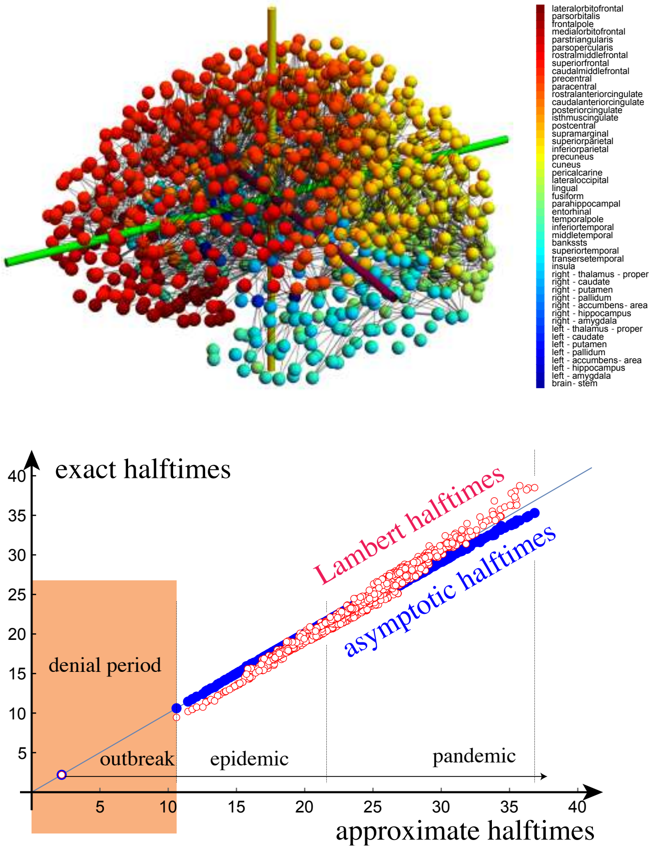


Figure 2: A brain connectome with 1015 nodes (top) and the computation of half-times for the dynamics (bottom) compared against numerical values (solid blue circles: nonlinear approximation, open red circles: Lambert temporal distance). Note the initial denial period, followed by a massive outbreak, leading to a global pandemic outbreak in the entire brain.

In Fig. 2 we compare the halftimes predicted by the Lambert distance and our nonlinear approximations against the numerical solution and find them both in very good qualitative agreement. A typical feature of the dynamics is an initial delay between the invasion of the first and second node when the disease appears in the posterior cingulate cortex. We call this the *denial period*, a time where the disease seems to be localised in a single region but is seeded, often below detection, to other region. As soon as the second node is seeded, the propagation turns from local outbreak to a systematic invasion and a global epidemic, not unlike the coronavirus pandemic observed in the first semester of 2021.

This type of analysis paves the way to understand different features of the disease such as the typical and systematic staging patterns known to occur in the brain [4], the role of clearance mechanisms [5], or multiple species of interacting proteins [6, 7]. We have developed open-access software package to study systematically *in silico* the effect of new drug treatments and understand the basic mechanisms underlying neurodegenerative diseases and dementia. More broadly, as an applied mathematics problem, these ideas can be applied to many other settings such as disease transmission [8] or the propagation of faults and failure in electric networks.

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PhD highlights

Ernest Latorre defended his PhD thesis, with the title *Active superelasticity in three-dimensional epithelia of controlled shape*, on 19 February 2021 under the supervision of ?Marino Arroyo Balaguer and ?Xavier Trepat Guixer, within the UPC doctoral program on Applied Mathematics. Currently he is at Principia.

Thesis summary. Fundamental processes in development and physiology are determined by the three-dimensional architecture of epithelial sheets. How these sheets deform and fold into complex structures has remained unclear, however, because their mechanical properties in three-dimensions have not been accessed experimentally. By combining measurements of epithelial tension, shape, and luminal pressure with mathematical modeling, here we show that epithelial cell sheets are active superelastic materials. We develop a new approach to produce massive arrays of epithelial domes with controlled basal shape and size. By measuring 3D deformations of the substrate and curvature of the dome we obtain a direct measurement of luminal pressure and epithelial tension. Observations over time-scales of hours allow us to map the epithelial tension-strain response, revealing a tensional plateau over several-fold areal strain reaching 300%. We show that these extreme nominal strains are accommodated by a highly heterogeneous stretching of individual cells, with barely deformed cells coexisting with others reaching 1000% areal strain, in seeming contradiction with the measured tensional uniformity. This phenomenology is reminiscent of superelasticity, a mechanical response generally attributed to microscopic material instabilities in metal alloys. We provide evidence that this instability is triggered in epithelial cells by a stretch-induced dilution of the actin cortex and rescued by the intermediate filament network. Finally, we implement a mathematical model that captures both the tension/strain relationship and strain heterogeneity. Our study unveils a new type of mechanical behavior –active superelasticity– that enables epithelial sheets to sustain extreme stretching under constant tension.

From the Introduction: "Many biological processes involve shape changes (deformation). We can find representative examples during morphogenesis or when performing routinely breathing, heart beating or peristaltic movements in mammals. Those deformations are ultimately caused by forces of active origin and thus are also governed by mechanics. This fact has been acknowledged by authors such as D'Arcy Thompson in his famous treatise *On growth and form*, where the author claimed that physical laws and more specifically mechanics is at the core of morphogenesis".

Highlighted publication. ?E. Latorre, ?S. Kale, ?L. Casares, ?M. Gómez-González, ?M. Uroz, ?L. Valon, ?R. V. Nair, ?E. Garreta, ?N. Montserrat, ?A. del Campo, ?B. Ladoux, ?M. Arroyo, ?X. Trepat: *Active superelasticity in three-dimensional epithelia of controlled shape*. Article in **Nature** 563 (2018) 203–208. See also **Nature**'s Technical review by ?Manuel Gómez-González, ?Ernest Latorre, ?Marino Arroyo, and ?Xavier Trepat.

Óscar Rivero defended his PhD thesis *Arithmetic applications of the Euler systems of Beilinson-Flach elements and diagonal cycles*, supervised by ?Víctor Rotger, on 12 February 2021 within the UPC doctoral program in Applied Mathematics. Currently he is in the group of David Loeffler at the University of Warwick and holding a Newton International Fellowship grant of the Royal Society.

Thesis summary. This thesis studies some applications to arithmetic problems of the so-called Euler systems, which are Galois cohomology classes that vary in a compatible way over towers of fields.

Following a fairly general philosophy introduced by Perrin–Riou, the image of these Euler systems under certain regulators allows us to recover the p-adic L-function associated with a Galois representation. In this thesis we focus mainly on the systems of Beilinson-Flach and diagonal cycles, although we also study others that share properties with the previous ones and that help us to better understand them. Let us mention that different works in the last decade have already succeeded, by means of these Euler systems, in proving new cases of the equivariant Birch and Swinnerton-Dyer conjecture, one of the great mathematical challenges of our times.

The arithmetic applications we discuss in this monograph are diverse: exceptional zeros, special value formulas, non-vanishing results, connections with Iwasawa theory ... The first chapters of the thesis study a phenomenon of exceptional zeros. Recall that the p-adic L-functions interpolate, over a certain region, values of the complex L-function multiplied by appropriate Euler factors. The vanishing of these factors often leads to interesting arithmetic phenomena. This, far from being fortuitous, supports an algebraic interpretation in terms of Selmer groups. For example, the cancellation at $s = 0$ of the Kubota-Leopoldt function is related to the fact that there is an extra p-unit in the corresponding component of the group of units, and its logarithm is related to the derivative of the p-adic L-function. This is one of Gross' best known conjectures.

Here we begin by studying the case of the adjoint representation of a weight 1 modular form. In this setting, we prove a conjecture of ?Darmon, ?Lauder, and ?Rotger that expresses the value of the derivative of the associated p-adic L-function in terms of a combination of logarithms of units and p-units in the field cut-out by the representation. The proof uses the theory of p-adic L-functions and improved p-adic L-functions, as well as Galois deformations.

In addition, we observe a phenomenon that complements this study. The p-adic L-functions that come out of it are the image of the Euler system of Beilinson-Flach elements by the Perrin-Riou morphism. These exceptional zeros are also observed at the level of Euler systems, and one can introduce the concept of derived class, that allows us to recover the $\mathcal{L}$-invariant that controls the arithmetic of the Galois representation. Not only that: with this notion of derivative we can give an alternative proof of the previous result by exploiting the geometry of these systems.

This first part of the thesis is complemented by two chapters where we work on the phenomenon of exceptional zeros for both elliptic units and diagonal cycles.

The final chapters delve into the study of other issues around Euler systems, and begin with the development of an Artin formalism at the level of cohomology classes. The most basic case is to consider a cuspidal eigenform f of weight 2 which is congruent to an Eisenstein series. The cohomology class associated with f gives rise, under appropriate conditions, to two different components modulo p. We suggest congruences relating each of them to expressions involving circular units. This uses, on the one hand, factorizations of p-adic L-functions and reciprocity laws; on the other hand, we recover some results of Fukaya–Kato developed during the study of Sharifi's conjectures.

Highlighted publication. ?O. Rivero, ?V. Rotger. Derived Beilinson–Flach elements and the arithmetic of the adjoint of a modular form. **Journal of the European Mathematical Society** JEMS (published online 2021-03-15). arXiv preprint.

Reviews

Mathematics and Machine Learning

by ?NL Editorial Team

Books

Large Networks and Graph Limits[1]

"Modern combinatorics is by no means an isolated subject in mathematics, but has many rich and interesting connections to almost every area of mathematics and computer science. The research presented in ?Lovász's book exemplifies this phenomenon by taking one of the most quintessentially combinatorial of objects –the finite graph– and through the process of taking limits of sequences of such graphs, reveals and clarifies connections to measure theory, analysis, statistical physics, metric geometry, spectral theory, property testing, algebraic geometry, and even Hilbert's tenth and seventeenth problems. Indeed, this book presents a wonderful opportunity for a student in combinatorics to explore other fields of mathematics, or conversely for experts in other areas of mathematics to become acquainted with some aspects of graph theory" (?Terence Tao).

"This review, of an important new book by the renowned mathematician László Lovász, is not a review written by an expert or for experts, but rather one that seeks to interpret and facilitate understanding of some of the book's highlights for a general mathematical audience. The reviewer learned a lot as he studied the book. It is exquisitely written, and thus suitable for self-study, as well as for use in graduate seminars or courses. One of the features is that the reader can come away with a broad-strokes understanding of the material by reading the 30-page Introduction. No key points are skipped. Fittingly, the author often reminds readers of this fact by referring them back to the Introduction at appropriate points, usually just before the theory is rigorously developed" (from MR3012035, by Anant P. ?Godbole).

Mathematics and Computation: A Theory Revolutionizing Technology and Science [2]

"Provides a broad, conceptual overview of computational complexity theory –the mathematical study of efficient computation. With important practical applications to computer science and industry, computational complexity theory has evolved into a highly interdisciplinary field, with strong links to most mathematical areas and to a growing number of scientific endeavors" (from the web page of the book).

"Avi Wigderson takes a sweeping survey of complexity theory, emphasizing the field's insights and challenges. He explains the ideas and motivations leading to key models, notions, and results. In particular, he looks at algorithms and complexity, computations and proofs, randomness and interaction, quantum and arithmetic computation, and cryptography and learning, all as parts of a cohesive whole with numerous cross-influences. Wigderson illustrates the immense breadth of the field, its beauty and richness, and its diverse and growing interactions with other areas of mathematics. He ends with a comprehensive look at the theory of computation, its methodology and aspirations, and the unique and fundamental ways in which it has shaped and will further shape science, technology, and society. For further reading, an extensive bibliography is provided for all topics covered. *Mathematics and Computation* is useful for undergraduate and graduate students in mathematics, computer science, and related fields, as well as researchers and teachers in these fields. Many parts require little background, and serve as an invitation to newcomers seeking an introduction to the theory of computation" (from Good reads)

Reinforcement Learning of Bimanual Robot Skills [3]

"The monograph by ?Adrià Colomé and ?Carme Torras is based on the first author's doctoral thesis. It conjugates the two research worlds of artificial intelligence and robotics in one original work on reinforcement learning for bimanual robots. The contents are effectively organized in two parts: (i) compliant redundant robot control, which is based on the well-known Jacobian-based CLIK [Closed Loop Inverse Kinematics] algorithm for solving inverse kinematics, along with an estimate of the external force and friction model; (ii) reinforcement learning with movement primitives, which is based on a new Dual REPS [Relative Entropy Policy Search] algorithm for fast learning of robot motion, along with a reduction of dimensionality of the motion parameterization" (from the Preface by ?Bruno Siciliano).

Data-Driven Computational Neuroscience: Machine Learning and Statistical Models [4]

"This book describes statistical and machine learning methods to build computational models learned from real-world neuroscience data. These methods cover supervised and unsupervised classification with probabilistic and non-probabilistic models, association discovery with probabilistic graphical models, and spatial statistics with point processes. Chapters with necessary basics of statistics are also included." (from the Preface).

Geometric Deep Learning: Grids, Groups, Graphs, Geodesics, and Gauges [5] (The "Erlangen Program" of Machine Learning).

"At the time of writing, the state of the field of deep learning is somewhat reminiscent of the field of geometry in the nineteenth century. There is a veritable zoo of neural network architectures for various kinds of data, but few unifying principles. As in times past, this makes it difficult to understand the relations between various methods, inevitably resulting in the reinvention and rebranding of the same concepts in different application domains. For a novice trying to learn the field, absorbing the sheer volume of redundant ideas is a true nightmare.

In this text, we make a modest attempt to apply the Erlangen Programme mindset to the domain of deep learning, with the ultimate goal of obtaining a systematisation of this field and 'connecting the dots'. We call this geometrisation attempt 'Geometric Deep Learning', and true to the spirit of Felix Klein, propose to derive different inductive biases and network architectures implementing them from first principles of symmetry and invariance. In particular, we focus on a large class of neural networks designed for analysing unstructured sets, grids, graphs, and manifolds, and show that they can be understood in a unified manner as methods that respect the structure and symmetries of these domains.

We believe this text would appeal to a broad audience of deep learning researchers, practitioners, and enthusiasts. A novice may use it as an overview and introduction to Geometric Deep Learning. A seasoned deep learning expert may discover new ways of deriving familiar architectures from basic principles and perhaps some surprising connections. Practitioners may get new insights on how to solve problems in their respective fields. With such a fast-paced field as modern machine learning, the risk of writing a text like this is that it becomes obsolete and irrelevant before it sees the light of day. Having focused on foundations, our hope is that the key concepts —–or, as Claude Adrien Helvétius put it, *"la connaissance de certains principes supplée facilement à la connaissance de certains faits"* " (from the Preface).

A thousand brains: A new theory of intelligence [6]
"Don't read this book at bedtime. Not that it's frightening. It won't give you nightmares. But it is so exhilarating, so stimulating, it'll turn your mind into a whirling maelstrom of excitingly provocative ideas—you'll want to rush out and tell someone rather than go to sleep. It is a victim of this maelstrom who writes the foreword, and I expect it'll show." (from Richard Dawkins' foreword). "Thinking about the universe and the uniqueness of intelligence is one of the reasons I wanted to study the brain. But there are plenty of other reasons right here on Earth. For example, understanding how the brain works has implications for medicine and mental health. Solving the brain's mysteries will lead to true machine intelligence, which will benefit all aspects of society in the same way that computers did, and it will lead to better methods of teaching our children. But ultimately, it comes back to our unique intelligence. We are the most intelligent species. If we want to understand who we are, then we have to understand how the brain creates intelligence. Reverse engineering the brain and understanding intelligence is, in my opinion, the most important scientific quest humans will ever undertake" (from the Final Thoughts chapter)

Papers

Fourier neural operator for parametric partial differential equations [7] "The classical development of neural networks has primarily focused on learning mappings between finite-dimensional Euclidean spaces. Recently, this has been generalized to neural operators that learn mappings between function spaces. For partial differential equations (PDEs), neural operators directly learn the mapping from any functional parametric dependence to the solution. Thus, they learn an entire family of PDEs, in contrast to classical methods which solve one instance of the equation. In this work, we formulate a new neural operator by parameterizing the integral kernel directly in Fourier space, allowing for an expressive and efficient architecture. We perform experiments on Burgers' equation, Darcy flow, and the Navier-Stokes equation (including the turbulent regime). Our Fourier neural operator shows state-of-the-art performance compared to existing neural network methodologies and it is up to three orders of magnitude faster compared to traditional PDE solvers." (Abstract)

Monte Carlo geometry processing: A grid-free approach to PDE-based methods on volumetric domains [8] "This paper explores how core problems in PDE-based geometry processing can be efficiently and reliably solved via grid-free Monte Carlo methods. Modern geometric algorithms often need to solve Poisson-like equations on geometrically intricate domains. Conventional methods most often mesh the domain, which is both challenging and expensive for geometry with fine details or imperfections (holes, self-intersections, etc.). In contrast, grid-free Monte Carlo methods avoid mesh generation entirely, and instead just evaluate closest point queries. They hence do not discretize space, time, nor even function spaces, and provide the exact solution (in expectation) even on extremely challenging models. More broadly, they share many benefits with Monte Carlo methods from photorealistic rendering: excellent scaling, trivial parallel implementation, view-dependent evaluation, and the ability to work with any kind of geometry (including implicit or procedural descriptions). We develop a complete 'black box' solver that encompasses integration, variance reduction, and visualization, and explore how it can be used for various geometry processing tasks. In particular, we consider several fundamental linear elliptic PDEs with constant coefficients on solid regions of $\mathbb{R}^n$. Overall we find that Monte Carlo methods significantly broaden the horizons of geometry processing, since they easily handle problems of size and complexity that are essentially hopeless for conventional methods." (Abstract)

Graph Laplacians, Riemannian manifolds and their machine-learning [9] "Graph Laplacians as well as related spectral inequalities and (co-)homology provide a foray into discrete analogues of Riemannian manifolds, providing a rich interplay between combinatorics, geometry and theoretical physics. We apply some of the latest techniques in data science such as supervised and unsupervised machine-learning and topological data analysis to the Wolfram database of some 8000 finite graphs in light of studying these correspondences. Encouragingly, we find that neural classifiers, regressors and networks can perform, with high efficiency and accuracy, a multitude of tasks ranging from recognizing graph Ricci-flatness, to predicting the spectral gap, to detecting the presence of Hamiltonian cycles, etc." (Abstract)

Neural Splines: Fitting 3D Surfaces with Infinitely-Wide Neural Networks [10] "We present Neural Splines, a technique for 3D surface reconstruction that is based on random feature kernels arising from infinitely-wide shallow ReLU networks. Our method achieves state-of-the-art results, outperforming recent neural network-based techniques and widely used Poisson Surface Reconstruction (which, as we demonstrate, can also be viewed as a type of kernel method). Because our approach is based on a simple kernel formulation, it is easy to analyze and can be accelerated by general techniques designed for kernel-based learning. We provide explicit analytical expressions for our kernel and argue that our formulation can be seen as a generalization of cubic spline interpolation to higher dimensions. In particular, the RKHS norm associated with Neural Splines biases toward smooth interpolants." (Abstract)

Discovering symbolic models from deep learning with inductive biases [11] "We develop a general approach to distill symbolic representations of a learned deep model by introducing strong inductive biases. We focus on Graph Neural Networks (GNNs). The technique works as follows: we first encourage sparse latent representations when we train a GNN in a supervised setting, then we apply symbolic regression to components of the learned model to extract explicit physical relations. We find the correct known equations, including force laws and Hamiltonians, can be extracted from the neural network. We then apply our method to a non-trivial cosmology example –a detailed dark matter simulation– and discover a new analytic formula which can predict the concentration of dark matter from the mass distribution of nearby cosmic structures. The symbolic expressions extracted from the GNN using our technique also generalized to out-of-distribution data better than the GNN itself. Our approach offers alternative directions for interpreting neural networks and discovering novel physical principles from the representations they learn."

The Convergence of AI code and Cortical Functioning – a Commentary [12] "Neural nets, one of the oldest architectures for AI programming, are loosely based on biological neurons and their properties. Recent work on language applications has made the AI code closer to biological reality in several ways. This commentary examines this convergence and, in light of what is known of neocortical structure, addresses the question of whether 'general AI' looks attainable with these tools" (Abstract).

Switch Transformers: Scaling to Trillion Parameter Models with Simple and Efficient Sparsity [13] "In deep learning, models typically reuse the same parameters for all inputs. Mixture of Experts (MoE) models defy this and instead select different parameters for each incoming example. The result is a sparsely-activated model –with an outrageous number of parameters– but a constant computational cost. However, despite several notable successes of MoE, widespread adoption has been hindered by complexity, communication costs, and training instability. We address these with the Switch Transformer. We simplify the MoE routing algorithm and design intuitive improved models with reduced communication and computational costs. Our proposed training techniques mitigate the instabilities, and we show large sparse models may be trained, for the first time, with lower precision (bfloat16) formats. We design models based off T5-Base and T5-Large (Raffel et al., 2019) to obtain up to 7x increases in pre-training speed with the same computational resources. These improvements extend into multilingual settings where we measure gains over the mT5-Base version across all 101 languages. Finally, we advance the current scale of language models by pre-training up to trillion parameter models on the 'Colossal Clean Crawled Corpus', and achieve a 4x speedup over the T5-XXL model." (Abstract)

Programmable Bootstrapping Enables Efficient Homomorphic Inference of Deep Neural Networks[14]

"In many cases, machine learning and privacy are perceived to be at odds. Privacy concerns are especially relevant when the involved data are sensitive. This paper deals with the privacy-preserving inference of deep neural networks. We report on first experiments with a new library implementing a variant of the TFHE fully homomorphic encryption scheme. The underlying key technology is the programmable bootstrapping. It enables the homomorphic evaluation of any function of a ciphertext, with a controlled level of noise. Our results indicate for the first time that deep neural networks are now within the reach of fully homomorphic encryption. Importantly, in contrast to prior works, our framework does not necessitate re-training the model" (Abstract).

Constructions in combinatorics via neural networks [15]

"We demonstrate how by using a reinforcement learning algorithm, the deep cross-entropy method, one can find explicit constructions and counterexamples to several open conjectures in extremal combinatorics and graph theory. Amongst the conjectures we refute are a question of Brualdi and Cao about maximizing permanents of pattern avoiding matrices, and several problems related to the adjacency and distance eigenvalues of graphs." (Abstract)
An interesting feature is that in some cases the learning algorithm does not produce directly a counterexample but graphs which are close to refuting the conjecture; these graphs have a special structure and give a very clear indication about where to search for counterexamples.

The Dawning of a New Era in Applied Mathematics [16]

"Can applied math become a unified subject with a small number of major components, like pure math? Can we have a reasonably unified curriculum to educate applied mathematicians? These questions have long been difficult to address. Looking back, it is clear that the situation just wasn't ripe…The situation has changed. With machine learning coming into the picture, all major components of applied math are now in place. This means that applied math is finally ready to become a mature scientific discipline. Yes new directions will continue to emerge, but there are reasons to believe that the fundamentals will more or less stay the same. These fundamentals are: (first-principle-based) modeling, learning, and algorithms."
"With such a program in place, applied math will become the foundation for interdisciplinary research. After all, modeling, learning, and algorithms are the fundamental components of all theoretical interdisciplinary research. The applied math program described above will help to systematize the training of students as well as the organization of interdisciplinary research programs. Should this become reality, it will be a turning point in the history of interdisciplinary research" (Extracted from the article).

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Quotations

?Richard Feynman (1961)

If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? I believe it is the *atomic hypothesis* (or the atomic *fact*, or whatever you wish to call it) that *all things are made of atoms—little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another.*

(Begins §I-1.2 of *The Feynman Lectures on Physics*: LoP-I, LoP-II, LoP-III).

?Eva Miranda (2020)

Poincaré himself was an example of polymath. Polymaths is what science needs now!

(8ECM: An Interview of Eva Miranda, 12 May 2020).

?Frank Wilczek (2021)

To experience the deep harmony between two different universes—the universe of beautiful ideas and the universe of physical behavior—was for me a kind of spiritual awakening. It became my vocation. I haven't been disappointed.

(In Chapter 3 of *Fundamentals: Ten Keys to Reality*. Penguin Press, 2021).

?Alan Turing (1938)

Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two faculties, which we may call intuition and ingenuity. The activity of the intuition consists in making spontaneous judgments which are not the result of conscious trains of reasoning... The exercise of ingenuity in mathematics consists in aiding the intuition through suitable arrangements of propositions, and perhaps geometrical figures or drawings. It is intended that when these are really well arranged validity of the intuitive steps which are required cannot seriously be doubted.

(In Chapter 11 of Turing's PhD thesis System of logic based on ordinals).

?John Ewing (1998)

Thirty-five years ago, when I arrived at Indiana University, a senior faculty member took me aside and gave me some advice. "Ewing," he said, "here are five things you ought to do in the next few years if you want to be a mathematician. Publish sensibly, discover constantly, travel extensively, talk widely, teach passionately. Do those things, and mathematics becomes a part of your life – not just a job, but a profession." That was good advice. (*Some Advice for Young Mathematicians*, Notices of the AMS, Volume **45**, Number 11, pdf)

?Avi Wigderson (2019)

An intuitive desire is to let $P \neq NP$ play the same role that the second law of thermodynamics plays in science: Scientists would be extremely wary to propose a model that violates it. One suggestion, by ?Scott Aaronson, is a stronger statement about the real world: *There are no physical means to solve NP-complete problems in polynomial time.*

(From §3.10, The nature and impact of NP-completeness, in *Mathematics and Computation*).

?László Lovász

A fundamental tool in the extremal theory of dense graphs is Szemerédi's Regularity Lemma, and this lemma turned out to be crucial for us as well. Graph limit theory, we hope, repaid some of this debt, by providing the shortest and most general formulation of the Regularity Lemma ("compactness of the graphon space").

(From the Preface of *Large networks and graph limits*).

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